In this chapter, you will learn to:

1. Write sample spaces.

2. Determine whether two events are mutually exclusive.

3. Use the Addition Rule.

4. Calculate probabilities using both tree diagrams and combinations.

5. Do problems involving conditional probability.

6. Determine whether two events are independent.

# 8.1 Sample Spaces and Probability

In this section, you will learn to:

1. Write sample spaces.

2. Calculate probabilities by examining simple events in sample spaces

If two coins are tossed, what is the probability that both coins will fall heads? The problem seems simple enough, but it is not uncommon to hear the incorrect answer 1/3. A student may incorrectly reason that if two coins are tossed there are three possibilities, one head, two heads, or no heads. Therefore, the probability of two heads is one out of three. The answer is wrong because if we toss two coins there are four possibilities and not three. For clarity, assume that one coin is a penny and the other a nickel. Then we have the following four possibilities.

HH HT TH TT

The possibility HT, for example, indicates a head on the penny and a tail on the nickel, while TH represents a tail on the penny and a head on the nickel.

It is for this reason, we emphasize the need for understanding sample spaces.

## SAMPLE SPACES

An act of flipping coins, rolling dice, drawing cards, or surveying people are referred to as a probability **experiment**.

A **sample space** of an experiment is the set of all possible outcomes.

***Example 1*** If a die is rolled, write a sample space.

***Solution:*** A die has six faces each having an equally likely chance of appearing. Therefore, the set of all possible outcomes S is

{1, 2, 3, 4, 5, 6}.

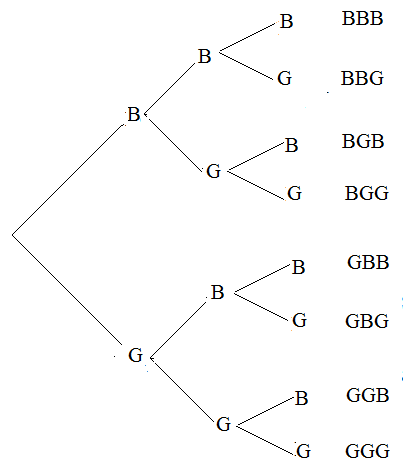
***Example 2*** A family has three children. Write a sample space.

***Solution:*** The sample space consists of eight possibilities.

{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}

The possibility BGB, for example, indicates that the first born is a boy, the second born a girl, and the third a boy.

We illustrate these possibilities with a tree diagram.



***Example 3*** Two dice are rolled. Write the sample space.

***Solution:*** We assume one of the dice is red, and the other green. We have the following 36 possibilities.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  | **Green** |  |  |  |
| **Red** | **1** | **2** | **3** | **4** | **5** | **6** |
| **1** | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) |
| **2** | (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) |
| **3** | (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) |
| **4** | (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) |
| **5** | (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) |
| **6** | (6, 1) | (6, 2) | (6, 3) | (6, 4) | (6, 5) | (6, 6) |

The entry (2, 5), for example, indicates that the red die shows a 2, and the green a 5.

## PROBABILITY

Now that we understand the concept of a sample space, we will define probability.

|  |
| --- |
| **Probability**  For a sample space S, and an outcome A of S, the following two properties are satisfied.  1. If A is an outcome of a sample space, then the probability of A, denoted by P(A), is between 0 and 1, inclusive.  0 ≤ P(A) ≤ 1  2. The sum of the probabilities of all the outcomes in S equals 1. |

The probability P(A) of an event A describes the chance or likelihood of that event occurring.

If P(A) = 0, event A is certain not to occur. If P(A) = 1, event A is certain to occur.

If P(A) = 0.5, then event A is equally likely to occur or not occur.

If we toss a fair coin that is equally likely to land on heads or tails, then P(Head) = 0.50.  
If the weather forecast says there is a 70% chance of rain today, then P(Rain) = 0.70, indicating is it more likely to rain than to not rain.

***Example 4*** If two dice, one red and one green, are rolled, find the probability that the red die shows a 3 and the green shows a six.

***Solution:*** Since two dice are rolled, there are 36 possibilities. The probability of each outcome, listed in Example 3, is equally likely.

Since (3, 6) is one such outcome, the probability of obtaining (3, 6) is 1/36.

The example we just considered consisted of only one outcome of the sample space.   
We are often interested in finding probabilities of several outcomes represented by an event.

An **event** is a subset of a sample space. If an event consists of only one outcome, it is called a **simple event.**

***Example 5*** If two dice are rolled, find the probability that the sum of the faces of the dice is 7.

***Solution:*** Let E represent the event that the sum of the faces of two dice is 7.

The possible cases for the sum to be equal to 7 are: (1, 6), (2,5), (3, 4), (4, 3), (5, 2),   
and (6, 1), so event E is

E = {(1, 6), (2,5), (3, 4), (4, 3), (5, 2), (6, 1)}

The probability of the event E is

P(E) = 6/36 or 1/6.

***Example 6*** A jar contains 3 red, 4 white, and 3 blue marbles. If a marble is chosen at random, what is the probability that the marble is a red marble or a blue marble?

***Solution:*** We assume the marbles are r1, r2, r3, w1, w2, w3, w4, b1, b2, b3. Let the event C represent that the marble is red or blue.

The sample space S = {r1, r2, r3, w1, w2, w3, w4, b1, b2, b3}

And the event C = {r1, r2, r3, b1, b2, b3}

Therefore, the probability of C,

P(C) = 6/10 or 3/5.

***Example 7*** A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn   
**without replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **without replacement**. That means that after a marble is drawn it is not replaced in the jar,   
and therefore is no longer available to select on the second draw.

***Solution:*** Since two marbles are drawn without replacement, the sample space consists of the following six possibilities.

S = {(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)}

Note that (1,1), (2,2) and (3,3) are not listed in the sample space. These outcomes are not possible when drawing without replacement, because once the first marble is drawn but not replaced into the jar, that marble is not available in the jar to be selected again on the second draw.

Let the event E represent that the sum of the numbers is five. Then

E = {(2, 3), (3, 2)}

Therefore, the probability of F is

P(E) = 2/6 or 1/3.

***Example 8*** A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is *at least* 4?

***Solution:*** The sample space, as in Example 7, consists of the following six possibilities.

S = {(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)}

Let the event F represent that the sum of the numbers is at least four. Then

F = {(1, 3), (3, 1), (2, 3), (3, 2)}

Therefore, the probability of F is

P(F) = 4/6 or 2/3.

***Example 9*** A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn   
**with replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **with replacement**. That means that after a marble is drawn it IS replaced in the jar, and therefore is available to select again on the second draw.

***Solution:*** When two marbles are drawn with replacement, the sample space consists of the following nine possibilities.

S = {(1,1), (1, 2), (1, 3), (2, 1), (2,2), (2, 3), (3, 1), (3, 2), (3,3)}

Note that (1,1), (2,2) and (3,3) are listed in the sample space. These outcomes are possible when drawing with replacement, because once the first marble is drawn and replaced, that marble is not available in the jar to be drawn again.

Let the event E represent that the sum of the numbers is four. Then

E = {(2, 3), (3, 2) }

Therefore, the probability of F is

P(E) = 2/9

Note that in Example 9 when we selected marbles with replacement, the probability has changed from Example 7 where we selected marbles without replacement.

***Example 10*** A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn   
**with replacement**, what is the probability that the sum of the numbers is *at least* 4?

***Solution:*** The sample space when drawing with replacement consists of the following nine possibilities.

S = {(1,1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3,3)}

Let the event F represent that the sum of the numbers is at least four. Then

F = {(1, 3), (3, 1), (2, 3), (3, 2), (2,2), (3,3)}

Therefore, the probability of F is

P(F) = 6/9 or 2/3.

Note that in Example 10 when we selected marbles with replacement, the probability is the same as in Example 8 where we selected marbles without replacement.

Thus sampling with or without replacement MAY change the probabilities, but may not, depending on the situation in the particular problem under consideration. We’ll re-examine the concepts of sampling with and without replacement in Section 8.3.

***Example 11*** One 6 sided die is rolled once. Find the probability that the result is greater than 4.

***Solution:*** The sample space consists of the following six possibilities in set S: S={1,2,3,4,5,6}

Let E be the event that the number rolled is greater than four: E={5,6}

Therefore, the probability of E is: P(E) = 2/6 or 1/3.

# 8.2 Mutually Exclusive Events and the Addition Rule

In this section, you will learn to:

1. Define compound events using union, intersection, and complement.

2. Identify mutually exclusive events

3. Use the Addition Rule to calculate probability for unions of events.

In the last chapter, we learned to find the union, intersection, and complement of a set.   
We will now use these set operations to describe events.

The **union** of two events E and F, EF, is the set of outcomes that are in E or in F or in both.

The **intersection** of two events E and F, EF, is the set of outcomes that are in both E and F.

The **complement** of an event E, denoted by Ec, is the set of outcomes in the sample space S that are not in E. It is worth noting that P(Ec) = 1 – P(E). This follows from the fact that if the sample space has n elements and E has k elements, then Ec has n – k elements. Therefore,

P(Ec) = = 1 – = 1 – P(E).

Of particular interest to us are the events whose outcomes do not overlap. We call these events mutually exclusive.

Two events E and F are said to be **mutually exclusive** if they do not intersect: E  F = .

Next we'll determine whether a given pair of events are mutually exclusive.

***Example 1*** A card is drawn from a standard deck. Determine whether the pair of events given below is mutually exclusive.

E = {The card drawn is an Ace}

F = {The card drawn is a heart}

***Solution:*** Clearly the ace of hearts belongs to both sets. That is

E  F = {Ace of hearts} .

Therefore, the events E and F are not mutually exclusive.

***Example 2*** Two dice are rolled. Determine whether the pair of events given below is mutually exclusive.

G = {The sum of the faces is six}

H = {One die shows a four}

***Solution:*** For clarity, we list the elements of both sets.

G = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)} and H = {(2, 4), (4, 2)}

Clearly, GH = {(2, 4), (4, 2)}.

Therefore, the two sets are not mutually exclusive.

***Example 3*** A family has three children. Determine whether the following pair of events are mutually exclusive.

M = {The family has at least one boy} N = {The family has all girls}

***Solution:*** Although the answer may be clear, we list both the sets.

M = {BBB, BBG, BGB, BGG, GBB, GBG, GGB} and N = {GGG}

Clearly, M N = 

Therefore, events M and N are mutually exclusive.

We will now consider problems that involve the union of two events.

Given two events, E, F, then finding the probability of EF, is the same as finding the probability that E will happen, or F will happen, or both will happen.

***Example 4*** If a die is rolled, what is the probability of obtaining an even number or a number greater than four?

***Solution:*** Let E be the event that the number shown on the die is an even number, and let F be the event that the number shown is greater than four.

The sample space S = {1, 2, 3, 4, 5, 6}. The event E = {2, 4, 6}, and event F = {5, 6}

We need to find P(E F).

Since P(E) = 3/6, and P(F) = 2/6, a student may say P(E F) = 3/6 + 2/6. This will be incorrect because the element 6, which is in both E and F has been counted twice, once as an element of E and once as an element of F. In other words, the set E F has only four elements and not five: set E F = {2,4,5,6}

Therefore, P(E F) = 4/6 and not 5/6.

This can be illustrated by a Venn diagram. We’ll use the Venn Diagram to re-examine Example 4 and derive a probability rule that we can use to calculate probabilities for unions of events.

The sample space S, the events E and F, and E  F are listed below.

S = {1, 2, 3, 4, 5, 6}, E = {2, 4, 6}, F = {5, 6}, and E  F = {6}.



The above figure shows S, E, F, and E  F.

Finding the probability of EF, is the same as finding the probability that E will happen, or F will happen, or both will happen.

If we count the number of elements n(E) in E, and add to it the number of elements n(F) in F, the points in both E and F are counted twice, once as elements of E and once as elements of F. Now if we subtract from the sum, n(E) + n(F), the number n(EF), we remove the duplicity and get the correct answer. So as a rule,

n(EF) = n(E) + n(F) – n(EF)

By dividing the entire equation by n(S), we get

= + – 

Since the probability of an event is the number of elements in that event divided by the number of all possible outcomes, we have

P(EF) = P(E) + P(F) – P(EF)

Applying the above for Example 4, we get

P(EF) = 3/6 + 2/6 – 1/6 = 4/6

This is because, when we add P(E) and P(F), we have added P(EF) twice. Therefore, we must subtract P(EF), once.

This gives us the general formula, called **the Addition Rule**, for finding the probability of the union of two events. Because event EF is the event that E will happen, OR F will happen, OR both will happen, we sometimes call this the **Addition Rule for OR Events.**It states

|  |
| --- |
| **Addition Rule:**  **P(EF) = P(E) + P(F) – P(EF)**  If, and only if, two events E and F are mutually exclusive, then EF =  and  P(EF) = 0, and we get P(EF) = P(E) + P(F) |

***Example 5*** If a card is drawn from a deck, use the addition rule to find the probability of obtaining an ace or a heart.

***Solution:*** Let A be the event that the card is an ace, and H the event that it is a heart.

Since there are four aces, and thirteen hearts in the deck,

P(A) = 4/52 and P(H) = 13/52.

Furthermore, since the intersection of two events consists of only one card, the ace of hearts, we now have:

P(AH) = 1/52

We need to find P(AH):

P(AH) = P(A) + P(H) – P(AH)

= 4/52 + 13/52 – 1/52 = 16/52.

***Example 6***  Two dice are rolled, and the events F and T are as follows:

F = {The sum of the dice is four} and T = {At least one die shows a three}

Find P(FT).

***Solution:*** We list F and T, and FT as follows:

F = {(1, 3), (2, 2), (3, 1)}

T = {(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)}

FT = {(1, 3), (3, 1)}

Since P(FT) = P(F) + P(T) – P(FT)

We have P(FT) = 3/36 + 11/36 – 2/36 = 12/36.

***Example 7*** Mr. Washington is seeking a mathematics instructor's position at his favorite community college in Cupertino. His employment depends on two conditions: whether the board approves the position, and whether the hiring committee selects him. There is a 80% chance that the board will approve the position, and there is a 70% chance that the hiring committee will select him. If there is a 90% chance that at least one of the two conditions, the board approval or his selection, will be met, what is the probability that Mr. Washington will be hired?

***Solution:*** Let A be the event that the board approves the position, and S be the event that Mr. Washington gets selected. We have,

P(A) = .80, P(S) = .70, and P(AS) = .90.

We need to find, P(AS).

The addition formula states that,

P(AS) = P(A) + P(S) – P(AS)

Substituting the known values, we get

.90 = .80 + .70 - P(AS)

Therefore, P(AS) = .60.

***Example 8*** The probability that this weekend will be cold is .6, the probability that it will be rainy is .7, and probability that it will be both cold and rainy is .5. What is the probability that it will be neither cold nor rainy?

***Solution:*** Let C be the event that the weekend will be cold, and R be event that it will be rainy. We are given that

P(C) = .6, P(R) = .7, P(CR) = .5

First we find P(CR) using the Addition Rule.

P(CR) = P(C) + P(R) – P(CR) = .6 + .7 – .5 = .8

Then we find P((CR)c) using the Complement Rule.

P((CR)c) = 1 – P(CR) = 1 – .8 = .2

We summarize this section by listing the important rules.

|  |
| --- |
| **The Addition Rule**  For Two Events E and F, P(EF) = P(E) + P(F) – P(EF)  **The Addition Rule for Mutually Exclusive Events**  If Two Events E and F are Mutually Exclusive, then P(EF) = P(E) + P(F)  **The Complement Rule**  If Ec is the Complement of Event E, then P(Ec) = 1 – P(E) |

# 8.3 Probability Using Tree Diagrams and Combinations

In this section, you will learn to:

1. Use probability tree diagrams to calculate probabilities

2. Use combinations to calculate probabilities

In this section, we will apply previously learnt counting techniques in calculating probabilities, and use tree diagrams to help us gain a better understanding of what is involved.

## USING TREE DIAGRAMS TO CALCULATE PROBABILITIES

We already used tree diagrams to list events in a sample space. Tree diagrams can be helpful in organizing information in probability problems; they help provide a structure for understanding probability. In this section we expand our previous use of tree diagrams to situations in which the events in the sample space are not all equally likely.

We assign the appropriate probabilities to the events shown on the branches of the tree.   
By multiplying probabilities along a path through the tree, we can find probabilities for “and” events, which are intersections of events.

We begin with an example.

***Example 1*** Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn with replacement, what is the probability that both marbles are red?

***Solution:*** Let E be the event that the first marble drawn is red, and let F be the event that the second marble drawn is red.

We need to find P(EF).

By the statement, "two marbles are drawn with replacement," we mean that the first marble is replaced before the second marble is drawn.

There are 7 choices for the first draw. And since the first marble is replaced before the second is drawn, there are, again, seven choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 49 ordered pairs. Of the 49 ordered pairs, there are 33 = 9 ordered pairs that show red on the first draw and, also, red on the second draw. Therefore,

P(EF) =

Further note that in this particular case

P(EF) = = .

giving us the result that in this example: **P(EF) = P(E) . P(F)**

***Example 2*** If in Example 1, the two marbles are drawn without replacement, then what is the probability that both marbles are red?

***Solution:*** By the statement, "two marbles are drawn without replacement," we mean that the first marble is not replaced before the second marble is drawn.

Again, we need to find P(EF).

There are, again, 7 choices for the first draw. And since the first marble is not replaced before the second is drawn, there are only six choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 42 ordered pairs. Of the 42 ordered pairs, there are 32 = 6 ordered pairs that show red on the first draw and red on the second draw. Therefore,

P(EF) =

Note that we can break this calculation down as

P(EF) = = .

Here 3/7 represents P(E), and 2/6 represents the probability of drawing a red on the second draw, given that the first draw resulted in a red.

We write the latter as P(red on the second | red on first) or P(F | E). The "|" represents the word "given" or “if”. This leads to the result that:

**P(EF) = P(E) . P(F | E)**

The is an important result, called the **Multiplication Rule,** which will appear again in later sections.

We now demonstrate the above results with a tree diagram.

***Example 3*** Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn without replacement, find the following probabilities using a tree diagram.

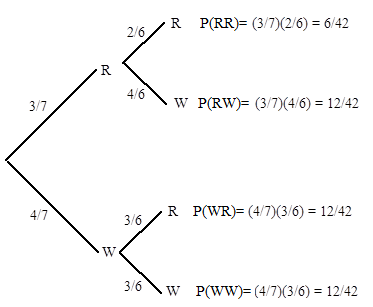
a. The probability that both marbles are red.

b. The probability that the first marble is red and the second white.

c. The probability that one marble is red and the other white.

***Solution:*** Let R be the event that the marble drawn is red, and let W be the event that the marble drawn is white.

We draw the following tree diagram.



a. The probability that both marbles are red is P(RR)=6/42

b. The probability that the first marble is red and the second is white is

P(RW)=12/42

c. For the probability that one marble is red and the other is white, we observe that this can be satisfied if the first is red and the second is white, **or** if the first is white and the second is red. The “or” tells us we’ll be using the Addition Rule from Section 7.2.

Furthermore events RW and WR are mutually exclusive events, so we use the form of the Addition Rule that applies to mutually exclusive events.

Therefore

P(one marble is red and the other marble is white)

= P(RW or WR)

= P(RW) + P(WR)

= 12/42 + 12/42 = 24/42.

## USING COMBINATIONS TO FIND PROBABILITIES

Although the tree diagrams give us better insight into a problem, they are not practical for problems where more than two or three things are chosen. In such cases, we use the concept of combinations that we learned in the last chapter. This method is best suited for problems where the order in which the objects are chosen is not important, and the objects are chosen without replacement.

***Example 4*** Suppose a jar contains 3 red, 2 white, and 3 blue marbles. If three marbles are drawn without replacement, find the following probabilities.

a. P(Two red and one white) b. P(One of each color)

c. P(None blue) d. P(At least one blue)

***Solution:*** Let us suppose the marbles are labeled as R1,R2,R3,W1,W2,B1,B2,B3.

a. P(Two red and one white)

Since we are choosing 3 marbles from a total of 8, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are 3C22C1 = 6 combinations consisting of 2 red and one white. Therefore,

P(Two red and one white) = = .

b. P(One of each color)

Again, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are 3C12C13C1 = 18 combinations consisting of one red, one white, and one blue. Therefore,

P(One of each color) = = .

c. P(None blue)

There are 5 non-blue marbles, therefore

P(None blue) = = = .

d. P(At least one blue)

By "at least one blue marble," we mean the following: one blue marble and two non-blue marbles, *OR* two blue marbles and one non-blue marble, *OR* all three blue marbles. So we have to find the sum of the probabilities of all three cases.

P(At least one blue) = P(1 blue, 2 non-blue) + P(2 blue, 1e non-blue) + P(3 blue)

P(At least one blue) = + +

P(At least one blue) = 30/56 + 15/56 + 1/56 = 46/56 = 23/28.

Alternately, we can use the fact that P(E) = 1 – P(Ec). If the event E = At least one blue, then Ec = None blue.

But from part c of this example, we have (Ec) = 5/28, so P(E) = 1 – 5/28 = 23/28.

***Example 5*** Five cards are drawn from a deck. Find the probability of obtaining two pairs, that is, two cards of one value, two of another value, and one other card.

***Solution:*** Let us first do an easier problem–the probability of obtaining a pair of kings and queens.

Since there are four kings, and four queens in the deck, the probability of obtaining two kings, two queens and one other card is

P(A pair of kings and queens) = 

To find the probability of obtaining two pairs, we have to consider all possible pairs.

Since there are altogether 13 values, that is, aces, deuces, and so on, there are 13C2 different combinations of pairs.

P(Two pairs) = 13C2 . = .04754

***Example 6*** A cell phone store receives a shipment of 15 cell phones that contains 8 iPhones and 7 Android phones. Suppose that 6 cell phones are randomly selected from this shipment.   
Find the probability that a randomly selected set of 6 cell phones consists of 2 iPhones and 4 Android phones.

***Solution:*** There are 8C2 ways of selecting 2 out of the 8 iPhones.

and 7C4 ways of selecting 4 out of the 7 Android phones

But altogether there are 15C6 ways of selecting 6 out of 15 cell phones.

Therefore we have

P(2 iPhones and 4 Android phones) = 

***Example 7*** One afternoon, a bagel store still has 53 bagels remaining: 20 plain, 15 poppyseed, and 18 sesame seed bagels. Suppose that the store owner packages up a bag of 9 bagels to bring home for tomorrow’s breakfast, and selects the bagels randomly. Find the probability that the bag contains 4 plain, 3 poppyseed, and 2 sesame seed.

***Solution:*** There are 20C4 ways of selecting 4 out of the 20 plain bagels,

and 15C3 ways of selecting 3 out of the 15 poppyseed bagels,

and 18C2 ways of selecting 2 out of the 18 sesame seed bagels.

But altogether there are 53C9 ways of selecting 9 out of the 53 bagels.   
P(4 plain, 3 poppyseed, and 2 sesame seed) = 





We end the section by solving a famous problem called the **Birthday Problem**.

***Example 8*** If there are 25 people in a room, what is the probability that at least two people have the same birthday?

***Solution:*** Let event E represent that at least two people have the same birthday.

We first find the probability that no two people have the same birthday.

We analyze as follows.

Suppose there are 365 days to every year. According to the multiplication axiom, there are 36525 possible birthdays for 25 people. Therefore, the sample space has 36525 elements. We are interested in the probability that no two people have the same birthday. There are 365 possible choices for the first person and since the second person must have a different birthday, there are 364 choices for the second, 363 for the third, and so on. Therefore,

P(No two have the same birthday) = =

Since P(at least two people have the same birthday) = 1 – P(No two have the same birthday),

P(at least two people have the same birthday) = 1 – = .5687

# 8.4 Conditional Probability

In this section, you will learn to:

1. recognize situations involving conditional probability

2. calculate conditional probabilities

Suppose a friend asks you the probability that it will snow today.

If you are in Boston, Massachusetts in the winter, the probability of snow today might be quite substantial. If you are in Cupertino, California in summer, the probability of snow today is very tiny, this probability is pretty much 0.

Let A = the event that it will snow today

B = the event that today you are in Boston in wintertime

C = the event that today you are in Cupertino in summertime

Because the probability of snow is affected by the location and time of year, we can’t just write P(A) for the probability of snow. We need to indicate the other information we know –location and time of year. We need to use **conditional probability**.

The event we are interested in is event A for snow. The other event is called the condition, representing location and time of year in this case.

We represent conditional probability using a vertical line | that means “if”, or “given that”, or “if we know that”. The event of interest appears on the left of the |. The condition appears on the right side of the |.

The probability it will snow given that (if) you are in Boston in the winter is represented by **P(A|B).** In this case, the condition is B.

The probability that it will snow given that (if) you are in Cupertino in the summer is represented by **P(A|C)**. In this case, the condition is C.

Now, let’s examine a situation where we can calculate some probabilities.

Suppose you and a friend play a game that involves choosing a single card from a well-shuffled deck. Your friend deals you one card, face down, from the deck and offers you the following deal: If the card is a king, he will pay you $5, otherwise, you pay him $1. Should you play the game?

You reason in the following manner. Since there are four kings in the deck, the probability of obtaining a king is 4/52 or 1/13. So, probability of not obtaining a king is 12/13.   
This implies that the ratio of your winning to losing is 1 to 12, while the payoff ratio is only $1 to $5. Therefore, you determine that you should not play.

But consider the following scenario. While your friend was dealing the card, you happened to get a glance of it and noticed that the card was a face card. Should you, now, play the game?

Since there are 12 face cards in the deck, the total elements in the sample space are no longer 52, but just 12. This means the chance of obtaining a king is 4/12 or 1/3. So your chance of winning is 1/3 and of losing 2/3. This makes your winning to losing ratio 1 to 2 which fares much better with the payoff ratio of $1 to $5. This time, you determine that you should play.

In the second part of the above example, we were finding the probability of obtaining a king knowing that a face card had shown. This is an example of **conditional probability**. Whenever we are finding the probability of an event E under the condition that another event F has happened, we are finding conditional probability.

The symbol P(E | F) denotes the problem of finding the probability of E given that F has occurred. We read P(E | F) as "the probability of E, given F."

***Example 1*** A family has three children. Find the conditional probability of having two boys and a girl given that the first born is a boy.

***Solution:*** Let event E be that the family has two boys and a girl, and F that the first born is a boy.

First, we the sample space for a family of three children as follows.

S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}

Since we know the first born is a boy, our possibilities narrow down to four outcomes: BBB, BBG, BGB, and BGG.

Among the four, BBG and BGB represent two boys and a girl.

Therefore, P(E | F} = 2/4 or 1/2.

***Example 2*** One six sided die is rolled once.

a. Find the probability that the result is even.

b. Find the probability that the result is even given that the result is greater than three.

***Solution:*** The sample space isS = {1,2,3,4,5,6}

Let event E be that the result is even and T be that the result is greater than 3.

a. P(E) = 3/6 because E = {2,4,6}

b. Because T = {4,5,6}, we know that 1, 2, 3 cannot occur; only outcomes 4, 5, 6 are possible. Therefore of the values in E, only 4, 6 are possible.

Therefore, P(E|T} = 2/3

***Example 3*** A fair coin is tossed twice.

a. Find the probability that the result is is two heads.

b. Find the probability that the result is two heads given that at least one head is obtained.

***Solution:*** The sample space isS = {HH, HT, TH, TT}

Let event E be that the two heads are obtained and F be at least one head is obtained

a. P(E) = 1/4 because E = {HH} and the sample space S has 4 outcomes.

b. F = {HH, HT, TH}. Since at least one head was obtained, TT did not occur.   
We are interested in the probability event E={HH} out of the 3 outcomes in the reduced sample space F.

Therefore, P(E|F} = 1/3

Let us now develop a formula for the conditional probability P(E | F).

Suppose an experiment consists of n equally likely events. Further suppose that there are m elements in F, and c elements in EF, as shown in the following Venn diagram.



If the event F has occurred, the set of all possible outcomes is no longer the entire sample space, but instead, the subset F. Therefore, we only look at the set F and at nothing outside of F. Since F has m elements, the denominator in the calculation of P(E | F) is m. We may think that the numerator for our conditional probability is the number of elements in E. But clearly we cannot consider the elements of E that are not in F. We can only count the elements of E that are in F, that is, the elements in EF. Therefore,

P(E | F) =

Dividing both the numerator and the denominator by n, we get

P(E | F) =

But c/n = P(EF), and m/n = P(F).

Substituting, we derive the following formula for P(E | F).

|  |
| --- |
| **Conditional Probability Rule**  For Two Events E and F, the probability of “E Given F” is  **P(E | F) =** |

***Example 4*** A single die is rolled. Use the above formula to find the conditional probability of obtaining an even number given that a number greater than three has shown.

***Solution:*** Let E be the event that an even number shows, and F be the event that a number greater than three shows. We want P(E | F).

E = {2, 4, 6} and F = {4, 5, 6}. Which implies, EF = { 4, 6}

Therefore, P(F) = 3/6, and P(EF) = 2/6

P(E | F) = = = .

***Example 5*** The following table shows the distribution by gender of students at a community college who take public transportation and the ones who drive to school.

|  |  |  |  |
| --- | --- | --- | --- |
|  | Male(M) | Female(F) | Total |
| Public Transportation(T) | 8 | 13 | 21 |
| Drive(D) | 39 | 40 | 79 |
| Total | 47 | 53 | 100 |

The events M, F, T, and D are self explanatory. Find the following probabilities.

a. P(D | M) b. P(F | D) c. P(M | T)

***Solution: Solution 1:*** Conditional probabilities can often be found directly from a contingency table. If the condition corresponds to only one row or only one column in the table, then you can ignore the rest of the table and read the conditional probability directly from the row or column indicated by the condition.

a. The condition is event M; we can look at only the “Male” column of the table and ignore the rest of the table: P(D | M) = .

b. The condition is event D; we can look at only the “Drive” row of the table and ignore the rest of the table: P(F | D) = .

c. The condition is event T; we can look at only the “Public Transportation” row of the table and ignore the rest of the table: P(M | T) = .

***Solution 2:*** We use the conditional probability formula P(E | F) = .

a. P(D | M) = = = .

b. P(F | D) = = = .

c. P(M | T) = = = .

***Example 6*** Given P(E) = .5, P(F) = .7, and P(EF) = .3. Find the following:

a. P(E | F) b. P(F | E).

***Solution:*** We use the conditional probability formula.

a. P(E | F) = == b. P(F | E) =  = .3/.5 = 3/5

***Example 7*** E and F are mutually exclusive events such that P(E) = .4, P(F) = .9. Find P(E | F).

***Solution:*** E and F are mutually exclusive, so P(EF) = 0.   
 Therefore P( E | F) =  = = 0.

***Example 8*** Given P(F | E) = .5, and P(EF) = .3. Find P(E).

***Solution:*** Using the conditional probability formula P(E | F) = , we get

P(F | E) = 

Substituting and solving:

.5 = or P(E) = 3/5

***Example 9*** In a family of three children, find the conditional probability of having two boys and a girl, given that the family has at least two boys.

***Solution:*** Let event E be that the family has two boys and a girl, and let F be the probability that the family has at least two boys. We want P(E | F).

We list the sample space along with the events E and F.

S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}

E = {BBG, BGB, GBB} and F = {BBB , BBG, BGB, GBB}

EF = {BBG, BGB, GBB}

Therefore, P(F) = 4/8, and P(EF) = 3/8, and

P(E | F) = = = .

***Example 10*** At a community college 65% of the students subscribe to Amazon Prime,   
 50% subscribe to Netflix, and 20% subscribe to both.   
If a student is chosen at random, find the following probabilities:

a. the student subscribes to Amazon Prime given that he subscribes to Netflix

b. the student subscribes to Netflix given that he subscribes to Amazon Prime

***Solution:*** Let A be the event that the student subscribes to Amazon Prime,   
and N be the event that the student subscribes to Netflix.

First identify the probabilities and events given in the problem.

P(student subscribes to Amazon Prime) = P(A) = 0.65

P(student subscribes to Netflix) = P(N) = 0.50

P(student subscribes to both Amazon Prime and Netflix) = P(A ∩N) = 0.20

Then use the conditional probability rule:

a. P(A | N) == =

b. P(N | A) == = .

# 8.5 Independent Events

In this section, you will:

1. define independent events

2. identify whether two events are independent or dependent

In the last section, we considered conditional probabilities. In some examples, the probability of an event changed when additional information was provided. This is not always the case. The additional information may or may not alter the probability of the event.   
  
In Example 1 we revisit the discussion at the beginning of the previous section and then contrast that with Example 2.

***Example 1*** A card is drawn from a deck. Find the following probabilities.

a. The card is a king. b. The card is a king given that the card is a face card.

***Solution:*** a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | The card is a face card), we reason as follows:

There are 12 face cards in a deck of cards. There are 4 kings in a deck of cards.

P(The card is a king | The card is a face card) = 4/12 = 1/3.

The reader should observe that in the above example,

P(The card is a king | The card is a face card) ≠ P(The card is a king)

In other words, the additional information, knowing that the card selected is a face card changed the probability of obtaining a king.

***Example 2*** A card is drawn from a deck. Find the following probabilities.

a. The card is a king. b. The card is a king given that a red card has shown.

***Solution:*** a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | A red card has shown), we reason as follows:

Since a red card has shown, there are only twenty six possibilities. Of the 26 red cards, there are two kings. Therefore,

P(The card is a king | A red card has shown) = 2/26 = 1/13.

The reader should observe that in the above example,

P(The card is a king | A red card has shown) = P(The card is a king)

In other words, the additional information, a red card has shown, did not affect the probability of obtaining a king.

Whenever the probability of an event E is not affected by the occurrence of another event F, and vice versa, we say that the two events E and F are **independent.** This leads to the following definition.

|  |
| --- |
| Two Events E and F are **independent** if and only if at least one of the following two conditions is true.  **1. P(E | F) = P(E) or 2. P(F | E) = P(F)**  If the events are not independent, then they are dependent.  If one of these conditions is true, then both are true. |

We can use the definition of independence to determine if two events are independent.  
  
We can use that definition to develop another way to test whether two events are independent.

Recall the conditional probability formula:

P(E | F) = 

Multiplying both sides by P(F), we get

P(EF) = P(E | F) P(F)

Now if the two events are independent, then by definition

P(E | F) = P(E)

Substituting, P(EF) = P(E) P(F)

We state it formally as follows.

|  |
| --- |
| **Test For Independence**  Two events E and F are independent if and only if  **P(EF) = P(E) P(F)** |

In the Examples 3 and 4, we’ll examine how to check for independence using both methods:

* Examine the probability of intersection of events to check whether P(EF) = P(E)P(F)
* Examine conditional probabilities to check whether P(E|F)=P(E) or P(F|E)=P(F)

We need to use only **one** of these methods. Both methods, if used properly, will always give results that are consistent with each other.

Use the method that seems easier based on the information given in the problem.

***Example 3*** The table below shows the distribution of color-blind people by gender.

|  |  |  |  |
| --- | --- | --- | --- |
|  | Male(M) | Female(F) | Total |
| Color-Blind(C) | 6 | 1 | 7 |
| Not Color-Blind(N) | 46 | 47 | 93 |
| Total | 52 | 48 | 100 |

where M represents male, F represents female, C represents color-blind, and   
N represents not color-blind. Are the events color-blind and male independent?

***Solution 1:*** According to the test for independence, C and M are independent if and only if   
P(CM) = P(C)P(M).

From the table: P(C) = 7/100, P(M) = 52/100 and P(CM) = 6/100

So P(C) P(M) = (7/100)(52/100) = .0364

which is **not** equal to P(CM) = 6/100 = .06

Therefore, the two events are not independent. We may say they are dependent.

***Solution 2:*** C and M are independent if and only if P(C|M) = P(C).

From the total column P(C) = 7/100 = 0.07

From the male column P(C|M) = 6/52= 0.1154

Therefore P(C|M) ≠ P(C), indicating that the two events are not independent.

***Example 4*** In a city with two airports, 100 flights were surveyed. 20 of those flights departed late.  
45 flights in the survey departed from airport A; 9 of those flights departed late.   
55 flights in the survey departed from airport B; 11 flights departed late.   
Are the events "depart from airport A" and "departed late" independent?

***Solution 1:*** Let A be the event that a flight departs from airport A, and L the event that a flight departs late. We have

P(AL) = 9/100, P(A) = 45/100 and P(L) = 20/100

In order for two events to be independent, we must have P(AL) = P(A) P(L)

Since P(AL) = 9/100 = 0.09

and P(A) P(L) = (45/100)(20/100) = 900/10000 = 0.09

the two events "departing from airport A" and "departing late" are independent.

***Solution 2:*** The definition of independent events states that two events are independent if P(E|F)=P(E).

In this problem we are given that

P(L|A) = 9/45= 0.2 and P(L) = 20/100 = 0.2

P(L|A) = P(L), so events "departing from airport A" and "departing late" are independent.

***Example 5*** A coin is tossed three times, and the events E, F and G are defined as follows:

E: The coin shows a head on the first toss.

F: At least two heads appear.

G: Heads appear in two successive tosses.

Determine whether the following events are independent.

a. E and F b. F and G c. E and G

***Solution:*** We list the sample space, the events, their intersections and the probabilities.

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

E = {HHH, HHT, HTH, HTT}, P(E) = 4/8 or 1/2

F = {HHH, HHT, HTH, THH}, P(F) = 4/8 or 1/2

G = {HHT, THH}, P(G) = 2/8 or 1/4

EF= {HHH, HHT, HTH}, P(EF) = 3/8

FG = {HHT, THH}, P(FG) = 2/8 or 1/4

EG = {HHT} P(EG) = 1/8

a. E and F will be independent if and only if P(EF) = P(E) P(F)

P(EF) =3/8 and P(E) P(F) = 1/2 . 1/2 = 1/4 .

Since 3/8 ≠ 1/4, we have P(EF) ≠ P(E) P(F).

Events E and F are not independent.

b. F and G will be independent if and only if P(FG) = P(F) P(G).

P(FG) = 1/4 and P(F) P(G) = 1/2 . 1/4 =1/8

Since 3/8 ≠ 1/4, we have P(FG) ≠ P(F) P(G).

Events F and G are not independent.

c. E and G will be independent if P(EG) = P(E) P(G)

P(EG) = 1/8 and P(E) P(G) = 1/2 . 1/4 =1/8

Events E and G are independent events because P(EG) = P(E) P(G)

***Example 6*** The probability that Jaime will visit his aunt in Baltimore this year is .30, and the probability that he will go river rafting on the Colorado river is .50. If the two events are independent, what is the probability that Jaime will do both?

***Solution:*** Let A be the event that Jaime will visit his aunt this year, and R be the event that he will go river rafting.

We are given P(A) = .30 and P(R) = .50, and we want to find P(AR).

Since we are told that the events A and R are independent,

P(AR) = P(A) P(R) = (.30)(.50) = .15

***Example 7*** Given P(B | A) = .4. If A and B are independent, find P(B).

***Solution:*** If A and B are independent, then by definition P(B | A) = P(B)

Therefore, P(B) = .4

***Example 8*** Given P(A) =.7, P(B| A) = .5. Find P(AB).

***Solution 1:*** By definition P(B | A) = 

Substituting, we have

.5 = 

Therefore, P(AB) = .35

***Solution 2:*** Again, start with P(B | A) = 

Multiplying both sides by P(A) gives

P(AB)= P(B| A) P(A) = (.5)(.7) = .35

Both solutions to Example 8 are actually the same, except that in Solution 2 we delayed substituting the values into the equation until after we solved the equation for P(AB). That gives the following result:

**Multiplication Rule for events that are NOT independent**

If events E and F are not independent

**P(EF) = P(E|F) P(F) and P(EF) = P(F|E) P(E)**

***Example 9*** Given P(A) =.5, P(AB ) = .7, if A and B are independent, find P(B).

***Solution:*** The addition rule states that

P(AB) = P(A) + P(B) – P(AB)

Since A and B are independent, P(AB) = P(A) P(B)

We substitute for P(AB) in the addition formula and get

P(AB) = P(A) + P(B) – P(A) P(B)

By letting P(B) = x, and substituting values, we get

.7 = .5 + x – .5x

.7 = .5 + .5x

.2 = .5x

.4 = x

Therefore, P(B) = .4