In this chapter, you will

1. examine exponential and logarithmic functions and their properties

2. identify exponential growth and decay functions and use them to model applications

3. use the natural base e to represent an exponential functions

4. use logarithmic functions to solve equations involving exponential functions

In this chapter we examine exponential and logarithmic functions. We will need these functions in the next chapter, when examining financial calculations.

This chapter is a new addition to this textbook. The California Community Colleges Curriculum Course Descriptor for Finite Mathematics (C-ID; https://c-id.net/descriptors.html, http://www.ccccurriculum.net/articulation/) now requires coverage of exponential and logarithmic functions in a Finite Mathematics course that is part of an Associate Degree for Transfer.

Students enrolling in Finite Mathematics typically are required to complete an Intermediate Algebra course or equivalent, as a prerequisite, so students have already been exposed to much of the material in this chapter. However many students require a review of this material, which is the basis for financial calculations based on compound interest in the following chapter. In addition, review of this material is particularly important at colleges where Finite Mathematics serves as a prerequisite for Business Calculus.

This book assumes students have mastered working with exponents, and properties of exponents; it focuses on review of exponential and logarithmic functions with an eye toward skills needed to use exponential growth and decay models for financial calculations and other business applications, as well as subsequent use in a course on Business Calculus. For the most part, financial applications are not stressed in this new chapter, as financial calculations are the focus of the following chapter.

Students requiring more extensive review than provided here should refer to their algebra textbook. In addition, textbooks for Algebra, College Algebra, and Precalculus from OpenStax are available to use free online and to download for free at <https://openstax.org/subjects/math>

The material in this chapter derives from the following sources:

David Lippman and Melonie Rasmussen, Open Text Bookstore, Precalculus: An Investigation of Functions, “[Chapter 4: Exponential and Logarithmic Functions](http://www.opentextbookstore.com/precalc/1.4/Chapter%204.doc),” licensed under a Creative Commons [CC BY-SA 3.0](http://creativecommons.org/licenses/by-sa/3.0/us/) license.

Precalculus textbook from OpenStax, which can be downloaded for free at <https://openstax.org/details/precalculus> , published under a Creative Commons Attribution License v4.0

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In addition, this chapter includes material written by Roberta Bloom, Mathematics Instructor at De Anza College.

# 5.1 Exponential Growth and Decay Models

*In this section, you will learn to*

*1. recognize and model exponential growth and decay*

*2. compare linear and exponential growth*

*3. distinguish between exponential and power functions*

## *COMPARING EXPONENTIAL AND LINEAR GROWTH*

Consider two social media sites which are expanding the number of users they have:

* Site A has 10,000 users, and expands by adding 1,500 new users each month
* Site B has 10,000 users, and expands by increasing the number of users by 10% each month.

The number of users for Site A can be modeled as linear growth. The number of users increases by a constant number, 1500, each month. If x = the number of months that have passed and y is the number of users, the number of users after x months is y = 10000+1500x.

For site B, the user base expands by a constant percent each month, rather than by a constant number. Growth that occurs at a constant percent each unit of time is called exponential growth.

We can look at growth for each site to understand the difference. The table compares the number of users for each site for 12 months. The table shows the calculations for the first 4 months only, but uses the same calculation process to complete the rest of the 12 months.

|  |  |  |
| --- | --- | --- |
| **Month** | **Users at Site A** | **Users at Site B** |
| 0 | 10000 | 10000 |
| 1 | 10000 + 1500 = 11500 | 10000 + 10% of 10000  = 10000 + 0.10(10000) =10000(1.10) = 11000 |
| 2 | 11500 + 1500 = 13000 | 11000 + 10% of 11000  = 11000 + 0.10(11000) =11000(1.10) = 12100 |
| 3 | 13000 + 1500 = 14500 | 12100 + 10% of 12100  = 12100 + 0.10(12100)  =12100(1.10) = 13310 |
| 4 | 14500 + 1500 = 16000 | 13310 + 10% of 13310  = 13310 + 0.10(13310)  =13310(1.10) = 14641 |
| 5 | 17500 | 16105 |
| 6 | 19000 | 17716 |
| 7 | 20500 | 19487 |
| 8 | 22000 | 21436 |
| 9 | 23500 | 23579 |
| 10 | 25000 | 25937 |
| 11 | 26500 | 28531 |
| 12 | 28000 | 31384 |

For Site B, we can re-express the calculations to help us observe the patterns and develop a formula for the number of users after x months.

Month 1: y = 10000(1.1) = 11000

Month 2: y = 11000(1.1) = 10000(1.1)(1.1)=**10000(1.1)2**= 12100

Month 3: y = 12100(1.1) = 10000(1.1) 2 (1.1)=**10000(1.1)3**= 13310

Month 4: y = 13310(1.1) = 10000(1.1)3 (1.1)=**10000(1.1)4**= 14641

By looking at the patterns in the calculations for months 2, 3, and 4, we can generalize the formula. After x months, the number of users y is given by the function **y = 10000(1.1)x**

## *USING EXPONENTIAL FUNCTIONS TO MODEL GROWTH AND DECAY*

In exponential growth, the value of the dependent variable y increases at a constant percentage rate as the value of the independent variable (x or t) increases. Examples of exponential growth functions include:

* the number of residents of a city or nation that grows at a constant percent rate.
* the amount of money in a bank account that earns interest if money is deposited at a single point in time and left in the bank to compound without any withdrawals.

In exponential decay, the value of the dependent variable y decreases at a constant percentage rate as the value of the independent variable (x or t) increases. Examples of exponential decay functions include:

* value of a car or equipment that depreciates at a constant percent rate over time
* the amount a drug that still remains in the body as time passes after it is ingested
* the amount of radioactive material remaining over time as a radioactive substance decays.

Exponential functions often model quantities as a function of time; thus we often use the letter t as the independent variable instead of x.

The table compares exponential growth and exponential decay functions:

|  |  |
| --- | --- |
| **Exponential Growth** | **Exponential Decay** |
| Quantity grows by a constant percent  per unit of time | Quantity decreases by a constant percent per unit of time |
| **y = abx**   * a is a positive number representing the initial value of the function when x = 0 * b is a real number that is greater than 1: b > 1 * the growth rate r is a positive number, r > 0 where b = 1+ r  (so that r = b−1) | **y = abx**   * a is a positive number  representing the initial value of  the function when x = 0 * b is a real number that is between  0 and 1: 0 < b < 1 * the decay rate r is a negative number, r < 0 where b = 1+ r  (so that r = b−1) |

In general, the domain of exponential functions is the set of all real numbers.   
The range of an exponential growth or decay function is the set of all positive real numbers.

In most applications, the independent variable, x or t, represents time. When the independent variable represents time, we may choose to restrict the domain so that independent variable can have only non-negative values in order for the application to make sense. If we restrict the domain, then the range is also restricted as well.

* For an exponential growth function y = abx with b>1 and a > 0,   
  if we restrict the domain so that x ≥ 0, then the range is y ≥ a.
* For an exponential decay function y = abx with 0<b<1 and a > 0,  
  if we restrict the domain so that x ≥ 0, then the range is 0 < y ≤ a.

***Example 1*** Consider the growth models for social media sites A and B, where x = number of months since the site was started and y = number of users.   
The number of users for Site A follows the linear growth model:   
 y = 10000+1500x.  
The number of users for Site B follows the exponential growth model:   
 y = 10000(1.1x)  
For each site, use the function to calculate the number of users at the end of the first year, to verify the values in the table. Then use the functions to predict the number of users after 30 months.

***Solution:*** Since x is measured in months, then x = 12 at the end of one year.

Linear Growth Model:

When x = 12 months, then y = 10000 + 1500(12) = 28,000 users  
When x = 30 months, then y = 10000 + 1500(30) = 55,000 users

Exponential Growth Model:

When x = 12 months, then y = 10000(1.112) = 31,384 users  
When x = 30 months, then y = 10000(1.130) =174,494 users

We see that as x, the number of months, gets larger, the exponential growth function grows large faster than the linear function (even though in Example 1 the linear function initially grew faster). This is an important characteristic of exponential growth: exponential growth functions always grow faster and larger in the long run than linear growth functions.

It is helpful to use function notation, writing y = f(t) = abt, to specify the value of t at which the function is evaluated.

***Example 2*** A forest has a population of 2000 squirrels that is increasing at the rate of 3% per year. Let t = number of years and y = f(t) = number of squirrels at time t.  
a. Find the exponential growth function that models the number of squirrels in the forest at the end of t years.   
b. Use the function to find the number of squirrels after 5 years and after 10 years

***Solution:*** a. The exponential growth function is y = f(t) = abt , where a = 2000 because the initial population is 2000 squirrels

The annual growth rate is 3% per year, stated in the problem. We will express this in decimal form as r = 0.03

Then b = 1+r = 1+0.03 = 1.03

Answer: The exponential growth function is y = f(t) = 2000(1.03t)

b. After 5 years, the squirrel population is y = f(5) = 2000(1.035) ≈ 2319 squirrels

After 10 years, the squirrel population is y = f(10) = 2000(1.0310) ≈ 2688 squirrels

***Example 3*** A large lake has a population of 1000 frogs. Unfortunately the frog population is decreasing at the rate of 5% per year.   
Let t = number of years and y = g(t) = the number of frogs in the lake at time t.  
a. Find the exponential decay function that models the population of frogs.  
b. Calculate the size of the frog population after 10 years.

***Solution:*** a. The exponential decay function is y = g(t) = abt, where a = 1000 because the initial population is 1000 frogs

The annual decay rate is 5% per year, stated in the problem. The words decrease and decay indicated that r is negative. We express this as r = −0.05 in decimal form.

Then, b = 1+ r = 1+ (−0.05) = 0.95

Answer: The exponential decay function is: y = g(t) = 1000(0.95t)

b. After 10 years, the frog population is y = g(10) = 1000(0.9510) ≈ 599 frogs

***Example 4*** A population of bacteria is given by the function y = f(t) = 100(2t), where t is time measured in hours and y is the number of bacteria in the population.  
a. What is the initial population?  
b. What happens to the population in the first hour?  
c. How long does it take for the population to reach 800 bacteria?

***Solution:*** a. The initial population is 100 bacteria. We know this because a = 100 and because at time t = 0, then f(0) = 100(20) = 100(1)=100

b. At the end of 1 hour, the population is y = f(1) = 100(21) = 100(2)=200 bacteria.  
The population has doubled during the first hour.

c. We need to find the time t at which f(t) = 800. Substitute 800 as the value of y:

y = f(t) =100(2t)

800 =100(2t)

Divide both sides by 100 to isolate the exponential expression on the one side

8 = 1(2t)

8 = 23, so it takes t = 3 hours for the population to reach 800 bacteria.

Two important notes about Example 4:

* In solving 8 = 2t, we “knew” that t is 3. But we usually can **not** know the value of the variable just by looking at the equation. Later we will use logarithms to solve equations that have the variable in the exponent.
* To solve 800 = 100(2 t), we divided both sides by 100 to isolate the exponential expression 2t. We can not multiply 100 by 2. Even if we write it as 800 =100(2)t, which is equivalent, we still can **not** multiply 100 by 2. The exponent applies **only** to the quantity immediately before it, so the exponent t applies only to the base of 2.

## *COMPARING LINEAR, EXPONENTIAL AND POWER FUNCTIONS*

To identify the type of function from its formula, we need to carefully note the position that the variable occupies in the formula.

**A** **linear function can be written in the form y = ax + b**.   
As we studied in chapter 1, there are other forms in which linear equations can be written, but linear functions can all be rearranged to have form y = mx + b.

**An exponential function** **has form y = abx**   
**The variable x is in the exponent.** The base b is a positive number.  
 If b>1, the function represents exponential growth.  
 If 0 < b < 1, the function represents exponential decay

**A power function has form y = cxp**   
**The variable x is in the base.** The exponent p is a non-zero number.

We compare three functions: • linear function y = f(x) = 2x

* exponential function y = g(x) = 2x
* power function y = h(x) = x2

|  |  |  |  |
| --- | --- | --- | --- |
| **x** | **y = f(x) =2x** | **y = g(x) =2x** | **y= h(x) =x2** |
| 0 | 0 | 1 | 0 |
| 1 | 2 | 2 | 1 |
| 2 | 4 | 4 | 4 |
| 3 | 6 | 8 | 9 |
| 4 | 8 | 16 | 16 |
| 5 | 10 | 32 | 25 |
| 6 | 12 | 64 | 36 |
| 10 | 20 | 1024 | 100 |
| Type of function | Linear y = mx+b | Exponential y = abx | Power y = cxp |
| How to recognize equation for this type of function. | all terms are first degree; m is slope;  b is the y intercept | base is a number b>0; the variable is in the exponent | variable is in the base; exponent is a number p≠0 |
|  | For equal intervals of change in x, y increases by a constant amount | For equal intervals of change in x, y increases by a constant ratio |  |

For the functions in the previous table: linear function y = f(x) = 2x, exponential function   
y = g(x) = 2x , and power function y = h(x) = x2 , if we restrict the domain to x ≥ 0 only,   
then all these functions are growth functions. When x ≥ 0, the value of y increases as the value of x increases.

The exponential growth function grows large faster than the linear and power functions, as x gets large. This is always true of exponential growth functions, as x gets large enough.

**Example 5** Classify the functions below as exponential, linear, or power functions.



**Solution:** The exponential functions are

 The variable is in the exponent; the base is the number b = 1.05  The variable is in the exponent; the base is the number b = 0.75

The linear functions are  and 

The power functions are

 The variable is the base; the exponent is a fixed number, p=3.

 The variable is the base; the exponent is a number, p=1/3.

The variable is the base; the exponent is a number, p = −2.

## *NATURAL BASE: e*

The number *e* is often used as the base of an exponential function. *e* is called the natural base.

***e* is approximately 2.71828**

*e* is an irrational number with an infinite number of decimals; the decimal pattern never repeats.

Section 6.2 includes an example that shows how the value of *e* is developed and why this number is mathematically important. Students studying Finite Math should already be familiar with the number *e* from their prerequisite algebra classes.

When *e* is the base in an exponential growth or decay function, it is referred to as **continuous growth or continuous decay**. We will use *e* in chapter 6 in financial calculations when we examine interest that compounds continuously.

**Any exponential function can be written in the form y = a*e*kx**

**k is called the continuous growth or decay rate.**

* + - If k > 0, the function represents exponential growth
    - If k< 0, the function represents exponential decay

**a is the initial value**

**We can rewrite the function in the form y = abx, where b=*e*k**

In general, if we know one form of the equation, we can find the other forms. For now, we have not yet covered the skills to find k when we know b. After we learn about logarithms later in this chapter, we will find k using natural log: k = ln b.

The table below summarizes the forms of exponential growth and decay functions.

|  |  |  |  |
| --- | --- | --- | --- |
|  | **y = abx** | **y = a(1+r)x** | **y = a*e*kx , k ≠ 0** |
| Initial value | a>0 | a>0 | a>0 |
| Relationship between b, r, k | b > 0 | b=1+ r | b = *e*k and k = ln b |
| Growth | b > 1 | r > 0 | k > 0 |
| Decay | 0 < b < 1 | r < 0 | k < 0 |

***Example 6*** The value of houses in a city are increasing at a continuous growth rate of 6% per year.

For a house that currently costs $400,000:

a. Write the exponential growth function in the form y=a*e*kx.

b. What would be the value of this house 4 years from now?

c. Rewrite the exponential growth function in the form y=abx.

d. Find and interpret r.

***Solution:*** a. The initial value of the house is a = $400000

The problem states that the **continuous** growth rate is 6% per year, so k = 0.06

The growth function is : y=400000*e*0**.**06x

b. After 4 years, the value of the house is y=400000*e*0**.**06 (4) = $508,500.

c. To rewrite y=400000e0.06x in the form y = abx, we use the fact that b=*e*k.

b=*e*0**.**06

b=1.06183657≈1.0618

y = 400000(1.0618)x

d. To find r, we use the fact that b=1+ r

b=1.0618

1+ r =1.0618

r =0.0618

The value of the house is **increasing** at an **annual rate** of 6.18%.

***Example 7*** Suppose that the value of a certain model of new car decreases at a continuous decay rate of 8% per year. For a car that costs $20,000 when new:

a. Write the exponential decay function in the form y=a*e*kx.

b. What would be the value of this car 5 years from now?

c. Rewrite the exponential decay function in the form y=abx.

d. Find and interpret r.

***Solution:*** a. The initial value of the car is a = $20000

The problem states that the **continuous** decay rate is 8% per year, so k = −0.08

The growth function is : y=20000*e−*0**.**08x

b. After 5 years, the value of the car is y=20000 *e−*0**.**08 (5) = $13,406.40.

c. To rewrite y=20000*e−*0**.**08x in the form y = abx, we use the fact that b=*e*k.

b=*e−*0**.**08

b=0.9231163464≈0.9231

y = 20000(0.9231)x

d. To find r, we use the fact that b=1+ r

b=0.9231

1+ r =0.9231

r =0.9231−1= −0.0769

The value of the car is **decreasing** at an **annual rate** of 7.69%.

# 5.2 Graphs and Properties of Exponential Growth and Decay Functions

*In this section, you will:*

*1. examine properties of exponential functions*

*2. examine graphs of exponential functions*

**An exponential function can be written in forms f(x) = abx = a(1 + r)x = a*e*kx**

**a is the initial value** because f(0) = a.   
In the growth and decay models that we examine in this finite math textbook, a > 0.

**b is often called the growth factor**. We restrict b to be positive ( b > 0) because even roots of negative numbers are undefined. We want the function to be defined for all values of x, but bx would be undefined for some values of x if b<0.

**r is called the growth or decay rate**. In the formula for the functions, we use r in decimal form, but in the context of a problem we usually state r as a percent.

**k is called the continuous growth rate or continuous decay rate**.

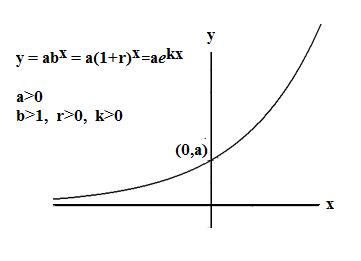
## *PROPERTIES OF EXPONENTIAL GROWTH FUNCTIONS*

The function y=f(x) = abx represents growth if b > 1 and a > 0.

The growth rate r is positive when b>1. Because b = 1+ r > 1, then r = b−1 > 0

The function y=f(x) = a*e*kx represents growth if k > 0 and a > 0.

The function is an increasing function; y increases as x increases.



Domain: { all real numbers} ; all real numbers can be input to an exponential function

Range: If a>0, the range is {positive real numbers}   
 The graph is always above the x axis.

Horizontal Asymptote: when b > 1, the horizontal asymptote is the negative x axis,   
as x becomes large negative. Using mathematical notation: as x → −∞, then y → 0.

The vertical intercept is the point (0,a) on the y-axis.   
There is no horizontal intercept because the function does not cross the x-axis.

## *PROPERTIES OF EXPONENTIAL DECAY FUNCTIONS*

The function y=f(x) = abx function represents decay if 0 < b < 1 and a > 0.

The growth rate r is negative when 0 < b<1. Because b = 1+ r < 1, then r = b−1 < 0.

The function y=f(x) = a*e*kx function represents decay if k < 0 and a > 0.

The function is a decreasing function; y decreases as x increases.

Domain: { all real numbers} ; all real numbers can be input to an exponential function

Range: If a>0, the range is {positive real numbers}  
 The graph is always above the x axis.

Horizontal Asymptote: when b < 1, the horizontal asymptote is the positive x axis  
 as x becomes large positive. Using mathematical notation: as x → ∞, then y → 0

The vertical intercept is the point (0,a) on the y-axis.   
There is no horizontal intercept because the function does not cross the x-axis.

The graphs for exponential growth and decay functions are displayed below for comparison

EXPONENTIAL GROWTH EXPONENTIAL DECAY

## 

## *AN EXPONENTIAL FUNCTION IS A ONE-TO-ONE FUNCTION*

Observe that in the graph of an exponential function, each y value on the graph occurs only once. Therefore, every y value in the range corresponds to only one x value. So, for any particular value of y, you can use the graph to see which value of x is the input to produce that y value as output. This property is called “**one-to-one**”.

Because for each value of the output y, you can uniquely determine the value of the corresponding input x, thus every exponential function has an inverse function.   
The inverse function of an exponential function is a logarithmic function, which we will investigate in the next section.

***Example 1*** x years after the year 2015, the population of the city of Fulton is given by the function y= f(x) = 35000(1.03x).   
x years after the year 2015, the population of the city of Greenville is given by the function y = g(x) = 80000(0.95x)  
Compare the graphs of these functions.

***Solution:*** The graphs below were created using computer graphing software.   
You can also graph these functions using a graphing calculator.

|  |  |
| --- | --- |
| Population of Fulton | Population of Greenville |
| y= f(x) = 35000(1.03x) | y= g(x) = 80000(0.95x) |
| Fulton’s population is increasing.  b = 1.03 >1 and r = 0.03 > 0  Exponential Growth | Greenville’s population is decreasing.  b = 0.95 < 1 and r = −0.05 < 0  Exponential Decay |
|  |  |
| **y-intercept:** (0, 35000)  The initial population in 2015 is 35000. | **y-intercept:** (0,80000)  The initial population in 2015 is 80000. |
| **Horizontal Asymptote:** The negative  x axis is the horizontal asymptote.  y → 0 as x → − ∞ | **Horizontal Asymptote:** The positive  x axis is the horizontal asymptote.  y → 0 as x → ∞ |

**Domain:** In general, the domain of both functions y= f(x) = 35000(1.03x) and   
y= g(x) = 80000(0.95x) is the set of all real numbers.

**Range:** The range of both functions is the set of positive real numbe rs. Both graphs always lie above the x-axis.

**Domain and Range in context of this problem:**The functions represent population size as a function of time ***after*** the year 2015 .   
We restrict the domain in this context, using the “practical domain” as the set of all  
non-negative real numbers: x≥0. Then we would consider only the portion of the graph that lies in the first quadrant.

* If we restrict the domain to x≥0 for the growth function y= f(x) = 35000(1.03x) , then the range for the population of Fulton is y ≥ 35,000
* If we restrict the domain to x≥0 for the decay function y= g(x) = 80000(0.95x), then the range for the population of Greenville is y ≤ 80,000.

# 5.3 Logarithms and Logarithmic Functions

*In this section you will learn*

*1. the definition of logarithmic function as the inverse of the exponential function*

*2. to write equivalent logarithmic and exponential expressions*

*3. the definition of common log and natural log*

*4. properties of logs*

*5. to evaluate logs using the change of base formula*

## THE LOGARITHM

Suppose that a population of 50 flies is expected to double every week, leading to a function of the form *f*(*x*) = 50(2)*x,* where *x* represents the number of weeks that have passed.  
When will this population reach 500?

Trying to solve this problem leads to 500 = 50(2)*x*

Dividing both sides by 50 to isolate the exponential leads to 10 = 2*x*

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation 2*x* = 10 above. We know that 23 = 8 and 24 = 16, so it is clear that *x* must be some value between 3 and 4 since *g*(*x*) = 2*x* is increasing. We could use technology to create a table of values or graph to better estimate the solution, but we would like to find an algebraic way to solve the equation.

We need an inverse operation to exponentiation in order to solve for the variable if the variable is in the exponent. As we learned in algebra class (prerequisite to this finite math course), the inverse function for an exponential function is a logarithmic function.

We also learned that an exponential function has an inverse function, because each output (y) value corresponds to only one input (x) value. The name given this property was “one-to-one”.

If you need to review the concept of an inverse function and how to determine if a function is one-to-one and has an inverse, you can refer to your algebra textbook or to College Algebra from Openstax available free online at <https://openstax.org/details/college-algebra>

Source: The material in this section of the textbook originates from David Lippman and Melonie Rasmussen, Open Text Bookstore, Precalculus: An Investigation of Functions, “[Chapter 4: Exponential and Logarithmic Functions](http://www.opentextbookstore.com/precalc/1.4/Chapter%204.doc),” licensed under a Creative Commons [CC BY-SA 3.0](http://creativecommons.org/licenses/by-sa/3.0/us/) license. The material here is based on material contained in that textbook but has been modified by Roberta Bloom, as permitted under this license.

**Logarithm**

The logarithm (base *b*) function, written log*b* (*x*), is the inverse of the exponential function (base *b*), *bx*.   
  **y = logb(x) is equivalent to by = x**

In general, the statement *ba* = *c* is equivalent to the statement log*b*(*c*) = *a*.

Note: The base *b* must be positive: *b*>0

**Inverse Property of Logarithms** Since the logarithm and exponential are inverses, it follows that:

 and 

Since log is a function, it is most correctly written as log*b* (*c*), using parentheses to denote function evaluation, just as we would with *f(c)*. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as log*b c*.

***Example 1*** Write these exponential equations as logarithmic equations:

a. 23 = 8 b. 52 = 25 c. 

***Solution:*** a. 23 = 8 can be written as a logarithmic equation as log2 (8) = 3

b. 52 = 25 can be written as a logarithmic equation as log5 (25) = 2

c.  can be written as a logarithmic equation as 

***Example 2*** Write these logarithmic equations as exponential equations:

a.  b. 

***Solution:*** a.  can be written as an exponential equation as   
b.  can be written as an exponential equation as 

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

***Example 3*** Solve log4 (*x*) = 2 for *x*.

***Solution:*** By rewriting this expression as an exponential, 42 = *x*, so *x* = 16

***Example 4*** Solve 2*x* = 10 for *x*.

***Solution:*** By rewriting this expression as a logarithm, we get *x* = log2 (10)

While this does define a solution, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful—often we really need a decimal approximation to the solution. Luckily, this is a task that calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases: base 10 and base *e*. Happily, this ends up not being a problem, as we’ll see soon that we can use a “change of base” formula to evaluate logarithms for other bases.

## COMMON AND NATURAL LOGARITHMS

The **common log** is the logarithm with base 10, and is typically written log (*x*).   
If the base is not indicated in the log function, then the base b used is b=10.

The **natural log** is the logarithm with base *e*, and is typically written ln (*x*).

Note that for any other base b, other than 10, the base must be indicated in the notation logb (x)

***Example 5*** Evaluate log(1000) using the definition of the common log.

***Solution:*** The table shows values of the common log

|  |  |  |
| --- | --- | --- |
| **number** | **number as exponential** | **log(number**) |
| 1000 | 103 | 3 |
| 100 | 102 | 2 |
| 10 | 101 | 1 |
| 1 | 100 | 0 |
| 0.1 | 10–1 | –1 |
| 0.01 | 10–2 | –2 |
| 0.001 | 10–3 | –3 |

To evaluate log(1000), we can say

*x* = log(1000)

Then rewrite the equation in exponential form using the common log base of 10

10*x* = 1000

From this, we might recognize that 1000 is the cube of 10, so

*x* = 3.

Alternatively, we can use the inverse property of logs to write

log10(103) = 3

***Example 6*** Evaluate log(1/1,000,000)

***Solution:*** To evaluate log(1/1,000,000), we can say

*x* = log(1/1,000,000) = log(1/106) = log (10−6)

Then rewrite the equation in exponential form: 10*x* = 10−6

Therefore x = −6

Alternatively, we can use the inverse property of logs to find the answer:  
log10(10−6) = −6

***Example 7*** Evaluate a. ln e5  b. .

***Solution:*** a. To evaluate ln e5, we can say  
*x* = ln e5

Then rewrite into exponential form using the natural log base of e  
e*x* = e*5*

Therefore x = 5

Alternatively, we can use the inverse property of logs to write ln(e5) = 5

b. To evaluate , we recall that roots are represented by fractional exponents

x =

Then rewrite into exponential form using the natural log base of e  
e*x* = e1/2

Therefore x = 1/2

Alternatively, we can use the inverse property of logs to write ln(e1/2) = 1/2

***Example 8*** Evaluate the following using your calculator or computer:

a. log 500 b. ln 500

***Solution:*** a. Using the LOG key on the calculator to evaluate logarithms in base 10, we evaluate LOG(500)

Answer: log 500 ≈ 2.69897

b. Using the LN key on the calculator to evaluate natural logarithms*,* we evaluate   
 LN(500)

Answer: ln 500 ≈ 6.214608

## SOME PROPERTIES OF LOGARITHMS

We often need to evaluate logarithms using a base other than 10 or *e*. To find a way to utilize the common or natural logarithm functions to evaluate expressions like log2(10), we need some additional properties.

**Properties of Logs: Exponential Property** : log*b*(*Aq*) = *q* log*b*(*A*)

The exponent property allows us to find a method for changing the base of a logarithmic expression.

**Properties of Logs: Change of Base**   
 for any bases b, c >0

To show why these properties are true, we offer proofs.

**Proof of Exponent Property**: log*b*(*Aq*) = *q*log*b*(*A*)

Since the logarithmic and exponential functions are inverses,

log*b*(*Aq*) = A

So 

Utilizing the exponential rule that states, we get



Then 

Again utilizing the inverse property on the right side yields the result  


**Proof of Change of Base Property:** for any bases b, c >0

Let log*b*(*A*) = *x*.

Rewriting as an exponential gives *bx = A*.

Taking the log base *c* of both sides of this equation gives log*c bx* = log*cA*.

Now utilizing the exponent property for logs on the left side,  
*x* log*c b* = log*c A*

Dividing, we obtain  which is the change of base formula.

## EVALUATING LOGARITHMS

With the change of base formula,  for any bases b, c >0,  
we can finally find a decimal approximation to our question from the beginning of the section.

***Example 9*** Solve 2*x* = 10 for *x*.

***Solution:*** Rewrite exponential equation 2*x* = 10 as a logarithmic equation

x = log 2(10)

Using the change of base formula, we can rewrite log base 2 as a logarithm of any other base. Since our calculators can evaluate natural log, we can choose to use the natural logarithm, which is the log base *e*:

Using our calculators to evaluate this,  = LN(10)/LN(2)  
This finally allows us to answer our original question from the beginning of this section:  
For the population of 50 flies that doubles every week, it will take approximately 3.32 weeks to grow to 500 flies.

***Example 10*** Evaluate log5(100) using the change of base formula.

***Solution:*** We can rewrite this expression using any other base.

Method 1: We can use natural logarithm base *e* with the change of base formula  
log5(100) =  = LN(100)/LN(5) ≈ 2.861

Method 2: We can use common logarithm base 10 with the change of base formula, log5(100) =  = LOG(100)/LOG(5) ≈ 2.861

We summarize the relationship between exponential and logarithmic functions

**Logarithms**

**The logarithm** (base *b*) function, written log*b*(*x*), is the inverse of the exponential function (base *b*), *bx*.

**y = logb(x) is equivalent to by = x**

In general, the statement *ba* = *c* is equivalent to the statement log*b*(*c*) = *a*.

Note: The base b must be positive: b>0

**Inverse Property of Logarithms** Since the logarithm and exponential are inverses, it follows that:

 and 

**Properties of Logs: Exponential Property:** log*b*(*Aq*) = *q* log*b*(*A*)   
  
**Properties of Logs: Change of Base**

 for any base b, c >0

The inverse, exponential and change of base properties above will allow us to solve the equations that arise in problems we encounter in this textbook. For completeness, we state a few more properties of logarithms

**Sum of Logs Property:** 

**Difference of Logs Property:** 

**Logs of Reciprocals:** 

**Reciprocal Bases:** 

Source: The material in this section of the textbook originates from David Lippman and Melonie Rasmussen, Open Text Bookstore, Precalculus: An Investigation of Functions, “[Chapter 4: Exponential and Logarithmic Functions](http://www.opentextbookstore.com/precalc/1.4/Chapter%204.doc),” licensed under a Creative Commons [CC BY-SA 3.0](http://creativecommons.org/licenses/by-sa/3.0/us/) license. The material here is based on material contained in that textbook but has been modified by Roberta Bloom, as permitted under this license.

# 5.4 Graphs and Properties of Logarithmic Functions

*In this section, you will:*

*1. examine properties of logarithmic functions*

*2. examine graphs of logarithmic functions*

*3. examine the relationship between graphs of exponential and logarithmic functions*

Recall that the exponential function  produces this table of values

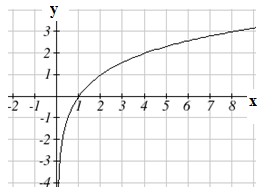
|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| f(x) | 1/8 | 1/4 | 1/2 | 1 | 2 | 4 | 8 |

Since the logarithmic function is an inverse of the exponential,  produces the table of values

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| x | 1/8 | 1/4 | 1/2 | 1 | 2 | 4 | 8 |
| g(x) | -3 | -2 | -1 | 0 | 1 | 2 | 3 |

In this second table, notice that

1. As the input increases, the output increases.
2. As input increases, the output increases more slowly.
3. Since the exponential function only outputs positive values, the logarithm can only accept positive values as inputs, so the domain of the log function is .
4. Since the exponential function can accept all real numbers as inputs, the logarithm can have any real number as output, so the range is all real numbers or .



Plotting the graph of g(x) = log2 (x) from the points in the table , notice that as the input values for x approach zero, the output of the function grows very large in the negative direction, indicating a vertical asymptote at *x* = 0.

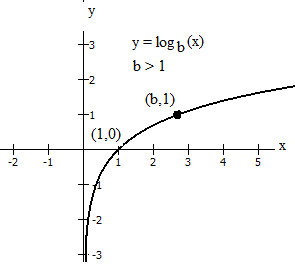
In symbolic notation we write

as 

and as

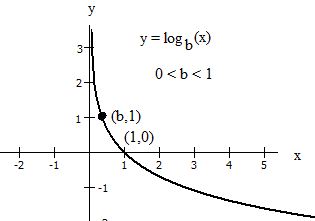
Source: The material in this section of the textbook originates from David Lippman and Melonie Rasmussen, Open Text Bookstore, Precalculus: An Investigation of Functions, “[Chapter 4: Exponential and Logarithmic Functions](http://www.opentextbookstore.com/precalc/1.4/Chapter%204.doc),” licensed under a Creative Commons [CC BY-SA 3.0](http://creativecommons.org/licenses/by-sa/3.0/us/) license. The material here is based on material contained in that textbook but has been modified by Roberta Bloom, as permitted under this license.

Graphically, in the function , b > 1, we observe the following properties:



* + The graph has a horizontal intercept at (1, 0)
  + The line x = 0 (the y-axis) is a vertical asymptote; as x →0+, y→ − ∞
  + The graph is increasing if b > 1
  + The domain of the function is *x* > 0, or 
  + The range of the function is all real numbers, or 

However if the base b is less than 1, 0 < b < 1, then the graph appears as below.  
This follows from the log property of reciprocal bases : 



* + The graph has a horizontal intercept at (1, 0)
  + The line x = 0 (the y-axis) is a vertical asymptote;; as x →0+, y→ ∞
  + The graph is decreasing if 0 < b < 1
  + The domain of the function is *x* > 0, or 
  + The range of the function is all real numbers, or 

When graphing a logarithmic function, it can be helpful to remember that the graph will pass through the points (1, 0) and (*b*, 1).

Finally, we compare the graphs of y = bx and y = logb (x), shown below on the same axes.

Because the functions are inverse functions of each other, for every specific ordered pair   
(h, k) on the graph of y = bx, we find the point (k, h) with the coordinates reversed on the graph of y = log b (x).

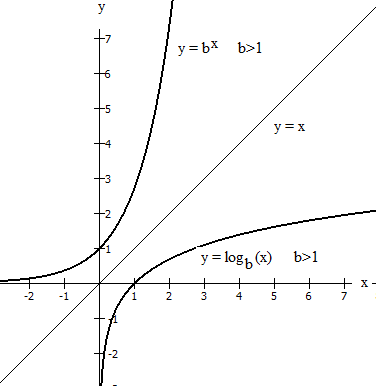
In other words, if the point with x = h and y = k is on the graph of y = bx, then the point with

x = k and y = h lies on the graph of y = logb (x)

The domain of y = bx is the range of y = logb (x)

The range of y = bx is the domain of y = logb (x)

For this reason, the graphs appear as reflections, or mirror images, of each other across the diagonal line y=x. This is a property of graphs of inverse functions that students should recall from their study of inverse functions in their prerequisite algebra class.



|  |  |  |
| --- | --- | --- |
|  | **y = bx, with b>1** | **y = logb (x) , with b>1** |
| Domain | all real numbers | all positive real numbers |
| Range | all positive real numbers | all real numbers |
| Intercepts | (0,1) | (1,0) |
| Asymptotes | Horizontal asymptote is  the line y = 0 (the x-axis)  As x→−∞, y →0 | Vertical asymptote is  the line x = 0 (the y axis)  As x→0+, y →−∞ |

Source: The material in this section of the textbook originates from David Lippman and Melonie Rasmussen, Open Text Bookstore, Precalculus: An Investigation of Functions, “[Chapter 4: Exponential and Logarithmic Functions](http://www.opentextbookstore.com/precalc/1.4/Chapter%204.doc),” licensed under a Creative Commons [CC BY-SA 3.0](http://creativecommons.org/licenses/by-sa/3.0/us/) license. The material here is based on material contained in that textbook but has been modified by Roberta Bloom, as permitted under this license.

# 5.5 Application Problems with Exponential and Logarithmic Functions

*In this section, you will:*

*1. review strategies for solving equations arising from exponential formulas*

*2. solve application problems involving exponential functions and logarithmic functions*

## STRATEGIES FOR SOLVING EQUATIONS THAT CONTAIN EXPONENTS

When solving application problems that involve exponential and logarithmic functions,   
we need to pay close attention to the position of the variable in the equation to determine the proper way solve the equation we investigate solving equations that contain exponents.

|  |
| --- |
| **Suppose we have an equation in the form : value = coefficient(base) exponent**  We consider four strategies for solving the equation:  **STRATEGY A:**  If the coefficient, base, and exponent are all known, we only need to evaluate the expression for coefficient(base) exponent to evaluate its value.  **STRATEGY B:**  If the variable is the coefficient, evaluate the expression for (base) exponent.  Then it becomes a linear equation which we solve by dividing to isolate the variable.  **STRATEGY C:** If the variable is in the exponent, use logarithms to solve the equation.  **STRATEGY D:** If the variable is not in the exponent, but is in the base, use roots to solve the equation. |

Below we examine each strategy with one or two examples of its use.

**STRATEGY A: If the coefficient, base, and exponent are all known, we only need to evaluate the expression for coefficient(base) exponent to evaluate its value.**

***Example 1***: Suppose that a stock’s price is rising at the rate of 7% per year, and that it continues to increase at this rate. If the value of one share of this stock is $43 now, find the value of one share of this stock three years from now.

***Solution***: Let y = the value of the stock after t years: y = ab t

The problem tells us that a = 43 and r = 0.07, so b = 1+ r = 1+ 0.07 = 1.07

Therefore, function is y = 43(1.07)t .

In this case we know that t = 3 years, and we need to evaluate y when t = 3.

At the end of 3 years, the value of this one share of this stock will be

y = 43(1.07)3= $52.68

**STRATEGY B: If the variable is the coefficient, evaluate the expression for (base) exponent.   
Then it becomes a linear equation which we solve by dividing to isolate the variable.**

***Example 2:*** The value of a new car depreciates (decreases) after it is purchased. Suppose that the value of the car depreciates according to an exponential decay model. Suppose that the value of the car is $12000 at the end of 5 years and that its value has been decreasing at the rate of 9% per year. Find the value of the car when it was new.

***Solution***: Let y be the value of the car after t years: y = ab t

r = −0.09 and b = 1+r = 1+(−0.09) = 0.91

The function is y = a(0.91) t

In this case we know that when t = 5, then y = 12000; substituting these values gives

12000 = a(0.91)5

We need to solve for the initial value a, the purchase price of the car when new.

First evaluate (0.91)5 ; then solve the resulting linear equation to find a.



 ; The car’s value was $19,230.77 when it was new.

**STRATEGY C: If the variable is in the exponent, use logarithms to solve the equation.**

***Example 3***: A national park has a population of 5000 deer in the year 2016. Conservationists are concerned because the deer population is decreasing at the rate of 7% per year.   
If the population continues to decrease at this rate, how long will it take until the population is only 3000 deer?

***Solution***: Let y be the number of deer in the national park t years after the year 2016: y = ab t

r = −0.07 and b = 1+r = 1+(−0.07) = 0.93 and the initial population is a = 5000

The exponential decay function is y = 5000(0.93) t

To find when the population will be 3000, substitute y = 3000

3000 = 5000(0.93) t

Next, divide both sides by 5000 to isolate the exponential expression  

Rewrite the equation in logarithmic form; then use the change of base formula to evaluate.

  years; After 7.039 years , there are 3000 deer.

Note: In Example 3, we needed to state the answer to several decimal places of precision to remain accurate. Evaluating the original function using a rounded value of t = 7 years gives a value that is close to 3000, but not exactly 3000.

 deer

However using t =7.039 years produces a value of 3000 for the population of deer

 deer

***Example 4*** A video posted on YouTube initially had 80 views as soon as it was posted.   
The total number of views to date has been increasing exponentially according to the exponential growth function y = 80e0.2t., where t represents time measured in days since the video was posted.Howmany days does it take until 2500 people have viewed this video?

***Solution***: Let y be the total number of views t days after the video is initially posted.  
We are given that the exponential growth function is y = 80e0.2t and we want to find the value of t for which y = 2500. Substitute y = 2500 into the equation and use natural log to solve for t.



Divide both sides by the coefficient, 80, to isolate the exponential expression.





Rewrite the equation in logarithmic form



Divide both sides by 0.04 to isolate t; then use your calculator and its natural log function to evaluate the expression and solve for t.





days

This video will have 2500 total views approximately 17 days after it was posted.

**STRATEGY D: If the variable is not in the exponent, but is in the base, we use roots to solve the equation.** *It is important to remember that we only use logarithms when the variable is in the exponent.*

***Example 5*** A statistician creates a website to analyze sports statistics. His business plan states that his goal is to accumulate 50,000 followers by the end of 2 years (24 months from now). He hopes that if he achieves this goal his site will be purchased by a sports news outlet. The initial user base of people signed up as a result of pre-launch advertising is 400 people.   
Find the monthly growth rate needed if the user base is to accumulate to 50,000 users at the end of 24 months.

***Solution***: Let y be the total user base t months after the site is launched.

The growth function for this site is y = 400(1+r)t;

We don’t know the growth rate r. We do know that when t = 24 months, then y = 50000.

Substitute the values of y and t; then we need to solve for r.



Divideboth sides by 400 to isolate b24 on one side of the equation



Because the variable in this equation is in the base, we use roots:



The website’s user base needs to increase at the rate of 22.28% per month in order to accumulate 50,000 users by the end of 24 months.

***Example 6*** A fact sheet on caffeine dependence from Johns Hopkins Medical Center states that the half life of caffeine in the body is between 4 and 6 hours. Assuming that the typical half life of caffeine in the body is 5 hours for the average person and that a typical cup of coffee has 120 mg of caffeine.

a. Write the decay function.

b. Find the hourly rate at which caffeine leaves the body.

c. How long does it take until only 20 mg of caffiene is still in the body?

https://www.hopkinsmedicine.org/psychiatry/research/bpru/docs/caffeine\_dependence\_fact\_sheet.pdf

***Solution***: a. Let y be the total amount of caffeine in the body t hours after drinking the coffee.

Exponential decay function y = abt models this situation.

The initial amount of caffeine is a = 120.

We don’t know b or r, but we know that the half- life of caffeine in the body is 5 hours. This tells us that when t = 5, then there is half the initial amount of caffeine remaining in the body.



Divide both sides by 120 to isolate the expression b5 that contains the variable.



The variable is in the base and the exponent is a number. Use roots to solve for b:



We can now write the decay function for the amount of caffeine (in mg.) remaining in the body t hours after drinking a cup of coffee with 120 mg of caffeine

y = f(t) = 120 (0.87) t

b. Use b = 1 + r to find the decay rate r. Because b = 0.87 < 1 and the amount of caffeine in the body is decreasing over time, the value of r will be negative.

0.87 = 1 + r

r = −0.13

The decay rate is 13%; the amount of caffeine in the body decreases by 13% per hour.

c. To find the time at which only 20 mg of caffeine remains in the body,   
substitute y = 20 and solve for the corresponding value of t.



Divide both sides by 120 to isolate the exponential expression.



Rewrite the expression in logarithmic form and use the change of base formula



hours

After 12.9 hours, 20 mg of caffeine remains in the body.

## EXPRESSING EXPONENTIAL FUNCTIONS IN THE FORMS y = abt and y = aekt

Now that we’ve developed our equation solving skills, we revisit the question of expressing exponential functions equivalently in the forms y = abt and y = a*e*kt

We’ve already determined that if given the form y = a*e*kt, it is straightforward to find b.

***Example 7*** For the following examples, assume t is measured in years.

a. Express y = 3500 *e* 0.25t in form y = abt and find the annual percentage growth rate.

b. Express y = 28000 *e*−0.32t in form y = abt and find the annual percentage decay rate.

***Solution***: a. Express y = 3500 *e* 0.25t in the form y = abt

y = a*e*kt = abt

a(*e*k)t = abt

Thus *e*k = b

In this example b = *e*0.25 ≈ 1.284

We rewrite the growth function as y = 3500(1.284t)

To find r, recall that b = 1+r  
1.284 = 1+r  
0.284 = r

The continuous growth rate is k = 0.25 and the annual percentage growth rate is 28.4% per year.

b. Express y = 28000 *e*−0.32t in the form y = abt

y = a*e*kt = abt

a(*e*k)t = abt

Thus *e*k = b

In this example b = *e*−0.32 ≈ 0.7261

We rewrite the growth function as y = 28000(0.7261t)

To find r, recall that b = 1+r  
0.7261 = 1+r  
0.2739 = r

The continuous decay rate is k = −0.32 and the annual percentage decay rate is 27.39% per year.

In the sentence, we omit the negative sign when stating the annual percentage decay rate because we have used the word “decay” to indicate that r is negative.

***Example 8*** a. Express y = 4200 (1.078)t in the form y = a*e*kt

b. Express y = 150 (0.73)t in the form y = a*e*kt

***Solution***: a. Express y = 4200 (1.078)t in the form y = a*e*kt

y = a*e*kt = abt

a(*e*k)t = abt

*e*k = b

*e*k = 1.078

Therefore k = ln 1.078 ≈ 0.0751

We rewrite the growth function as y = 3500*e* 0.0751t

b. Express y =150 (0.73)t in the form y = a*e*kt

y = a*e*kt = abt

a(*e*k)t = abt

*e*k = 0.73

*e*k = 0.73

Therefore k = ln 0.73 ≈ −0.3147

We rewrite the growth function as y = 150 *e*−0.3147t

## AN APPLICATION OF A LOGARITHMIC FUNCTON

Suppose we invest $10,000 today and want to know how long it will take to accumulate to a specified amount, such as $15,000. The time t needed to reach a future value y is a logarithmic function of the future value: t = g(y)

***Example 9*** Suppose that Vinh invests $10000 in an investment earning 5% per year.   
He wants to know how long it would take his investment to accumulate to $12000, and how long it would take to accumulate to $15000.

***Solution***: We start by writing the exponential growth function that models the value of this investment as a function of the time since the $10000 is initially invested

y = 10000(1.05)t

We divide both sides by 10000 to isolate the exponential expression on one side.



Next we rewrite this in logarithmic form to express time as a function of the accumulated future value. We’ll use function notation and call this function g(y).



Use the change of base formula to express t as a function of y using natural logarithm:



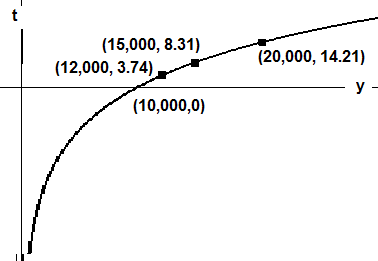
We can now use this function to answer Vinh’s questions.

To find the number of years until the value of this investment is $12,000, we substitute y = $12,000 into function g and evaluate t:

 years

To find the number of years until the value of this investment is $15,000, we substitute y = $15,000 into function g and evaluate t:

 years

Before ending this section, we investigate the graph of the function  . We see that the function has the general shape of logarithmic functions that we examined in section 5.4. From the points plotted on the graph, we see that function g is an increasing function but it increases very slowly.  
  
 

If we consider just the function  , then the domain of function would be y > 0, all positive real numbers, andthe range for t would be all real numbers.

In the context of this investment problem, the initial investment at time t = 0 is y =$10,000. Negative values for time do not make sense. Values of the investment that are lower than the initial amount of $10,000 also do not make sense for an investment that is increasing in value.

Therefore the function and graph as it pertains to this problem concerning investments has   
domain y ≥ 10,000 and range t ≥ 0.

The graph below is restricted to the domain and range that make practical sense for the investment in this problem.

